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On steady non-breaking downstream waves and the wave resistance – Stokes' method

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In this work, we have obtained explicit analytical formulae expressing the wave resistance of a two-dimensional body in terms of geometric parameters of nonlinear downstream waves. The formulae have been constructed in the form of high-order asymptotic expansions in powers of the wave amplitude with coefficients depending on the mean depth. To obtain these expansions, the second Stokes method has been used. The analysis represents the next step of the research carried out in Maklakov & Petrov (*J. Fluid Mech.*, vol. 776, 2015, pp. 290–315), where the properties of the waves have been computed by a numerical method of integral equations. In the present work, we have derived a quadratic system of equations with respect to the coefficients of the second Stokes method and developed an effective computer algorithm for solving the system. Comparison with previous numerical results obtained by the method of integral equations has been made.

Key words: channel flow, surface gravity waves, waves/free-surface flows

1. Introduction

Recently, Maklakov & Petrov (2015*a*, hereinafter referred to as paper 1) deduced several exact formulae for the wave resistance of a two-dimensional body that moves horizontally at a constant speed in a channel of finite depth. The formulae were derived under assumptions that in the body-fixed reference frame the flow is steady and irrotational and the capillarity effect on the free surface is negligibly small. It was demonstrated that to compute the wave resistance, it is enough to know the wave properties far downstream of the body. If, for example, the length, mean depth and amplitude of the waves are given, then the wave resistance can be determined. Although in paper 1 the computations of the waves were carried out only numerically by the method of self-generating converging meshes suggested in Maklakov (2002), at the end of the paper, it was indicated that by using the obtained results, it would be possible to deduce analytical formulae which connect the wave resistance with the wavelength, wave depth and amplitude. One of the ways to do so is the use of the second Stokes method, which has been applied in the presented paper.

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Stokes (1847, 1880) proposed two analytical methods of calculating the form of steady, irrotational waves of finite amplitude on water of finite depth. The first method (see Stokes 1847) involves the expansion of the complex velocity potential $w = \phi + \mathrm{i}\psi$ as a function of the space complex coordinate $z = x + \mathrm{i}y$ in powers of a small perturbation parameter responsible for the wave steepness. Later, he saw that the calculations would be simplified by expressing z as a function of w (see Stokes 1880). This is the second Stokes method, where the complex potential w is taken as an independent variable.

In the presented paper, we apply the second Stokes method to construct high-order analytical expansions in powers of the dimensionless wave amplitude a for the wave resistance and other wave properties. The coefficients of these expansions depend only on the dimensionless mean depth h_a. The reference length for non-dimensionalization is $\lambda/(2\pi)$, where λ is the wavelength. The application of the method requires determining the so-called Stokes coefficients, which are the Fourier coefficients for the function $z(w)$. In the literature, there is a considerable number of works devoted to this subject.

Stokes (1880) computed the solution to $O(b^3)$ for finite depth and to $O(b^5)$ for the particular case of infinite depth, where b is the first Fourier coefficient in the expansion of $z(w)$. Wilton (1914) carried the infinite depth computation up to $O(b^{10})$, but, as was noticed by Schwartz (1974), Wilton's expansions have errors starting with his eighth-order results. De (1955) has published a fifth-order solution for finite depth. It is to be noted that De's expansions are in powers of b with coefficients depending on the mean depth h_a.

It is worth mentioning also the work by Fenton (1985), although he used not the second but the first Stokes method. Fenton developed the fifth-order theory for waves of finite depth, and in his investigation the expansions are in powers of a with coefficients which are functions of h_a. In the presented paper, the expansions for the wave properties are of the same structure, and because exact asymptotic expansions are independent of the method by which they have been obtained, we have a good opportunity for comparison.

It seems that the works by Wilton (1914), De (1955) and Fenton (1985) involve the order of solution which is the practical limit of hand calculations. The first computer algorithm was developed by Schwartz (1974). As in Stokes (1880), Schwartz used the boundary condition of constant pressure on the free surface and obtained a cubic system of equations with respect to the Stokes coefficients. Expanding the latter in powers of a certain perturbation parameter, the algorithm allowed Schwartz to find the coefficients sequently. Schwartz used initially the first Stokes coefficient b as the perturbation parameter and demonstrated that at given values of b, the solution is not always unique. He showed also that this deficiency disappears by replacing b with the dimensionless wave amplitude a. The computations for finite depth were carried out by Schwartz up to $O(b^{70})$ and $O(a^{48})$. For the special case of infinite depth, the solution was found up to $O(a^{117})$. For this particular case, Schwartz presented also explicit analytical expansions of the Stokes coefficients up to $O(a^9)$. In these expansions, the coefficients are rational numbers, which were obtained by recognizing repeating patterns in the computer-produced decimals.

Cokelet (1977) applied the Schwartz algorithm with another perturbation parameter, suggested by Longuet-Higgins (1975), and made an extensive tabulation of the wave properties for different depth-to-wavelength ratios. The highest order of the perturbation parameter was 110. Rather recently, Dallaston & McCue (2010), using a modern desktop computer and exact arithmetics, which eliminates any loss of accuracy due to accumulation of round-off error, reproduced the Schwartz and Cokelet computations for the infinite depth case to an order of 300.

Thus the computer algorithms applied in the papers by Schwartz (1974), Cokelet (1977) and Dallaston & McCue (2010) are based on solving a cubic system of equations for the Stokes coefficients. Longuet-Higgins (1978) was the first to notice that after some transformations, the cubic system can be reduced to a quadratic one. He derived such quadratic systems both for the infinite and finite depth cases and developed for infinite depth a computer algorithm for finding expansions of the coefficients in powers of any perturbation parameter.

For infinite depth, analogous systems of quadratic equations were deduced later by Balk (1996) and Petrov (2009) from the Hamilton variational principle with the Lagrangian $L = T - V$, where T and V are the kinetic and potential energies of one wave period. For finite depth, a quadratic system of equations was deduced also in the paper by Dyachenko, Zakharov & Kuznetsov (1996) on the basis of a combination of the canonical formalism, introduced by Zakharov (1968), and conformal mapping of the flow region onto a horizontal strip.

It is to be noted that in the works by Longuet-Higgins (1978) and Dyachenko *et al.* (1996), the quadratic systems of equations for the case of finite depth were deduced but not solved. In Dyachenko *et al.* (1996), the authors confined themselves to expressing the second Fourier coefficient in terms of the first one and finding the second approximation for the dispersion relation. Longuet-Higgins (1978, p. 262) noticed that for finite depth the system deduced by him is not pure quadratic and developed the computer algorithm only for infinite depth.

In this work, we consider the steady wave train generated by a moving body on water of finite depth, and to compute the properties of the downstream waves, we derive also a system of quadratic equations with respect to the Stokes coefficients. To do so, we make use of the Luke variational principle (see Luke 1967) and demonstrate that for steady periodic waves the principle has a pure geometric interpretation, in the sense that the functional to be varied includes only geometric properties of periodic domains which are candidates for the gravity wave domain to be found. So the problem of steady periodic gravity waves can be reformulated in pure geometric terms (see § 2).

In § 3, by making use of this geometric interpretation, we obtain a compact system of pure quadratic equations with respect to the Stokes coefficients and one complementary parameter, which is the Bernoulli constant as it was defined in Schwartz (1974) and Cokelet (1977). It is worthwhile to notice that, formally, the same system can be obtained from the Hamilton variational principle (see Maklakov & Petrov 2015*b*), but for finite depth the Lagrangian $L = T - V$ should be extended to provide the fixed mean depth h_a. In this case, the complementary parameter, mentioned above, contains a Lagrange multiplier, and to prove that this parameter coincides with the Bernoulli constant is not a simple task.

In § 4, we describe an algorithm for finding the Stokes coefficients and in § 5, using some formulae from paper 1, derive asymptotic expansions for the wave resistance and other wave properties in powers of the wave amplitude a with coefficients depending on the mean depth h_a.

We should emphasize that among numerous works devoted to computing the progressive finite depth waves by the Stokes methods, there is only one in which exact analytical expansions are expressed in terms of the physical parameters, a and h_a. This is the work of Fenton (1985), where the maximum order of the expansions is five. The same maximum order, five, was achieved by De (1955), but instead of a,

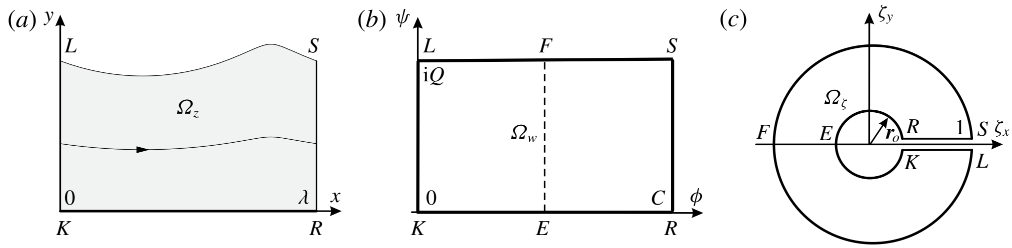

FIGURE 1. (*a*) One period of a λ-periodic domain in the physical z-plane, (*b*) plane of the complex potential $w=\phi+\mathrm{i}\psi$, (*c*) parametric ζ-plane.

De used the first Stokes coefficient b, whose physical meaning is not quite clear. To obtain the order of expansions higher than five, all authors specified the parameter responsible for the wave depth as a number, rational or real (see, e.g. Schwartz 1974; Cokelet 1977; Dallaston & McCue 2010). This leads to an asymptotic series in powers of a certain perturbation parameter with coefficients which are also numbers. To get the latter, the authors need to run some computer programme, so such approaches cannot be considered to be purely analytical. The main goal of the presented paper is to obtain exact analytical expansions of order much higher than five by making use of the second Stokes method.

It is also to be noted that the results for the wave resistance obtained in the paper are correct not only for a body that moves under a free surface but also for a plate planing on a water surface without spray formation, for a bump on a horizontal bottom or for a free-surface flow over a system of concentrated singularities, such as vortices and doublets. So the results are independent of the type of flow disturbance under the assumption that on the free surface there are no wavebreaking and sprays. The only requirement is that the levels of the bottom far upstream and far downstream of the disturbance are equal.

2. Geometric interpretation of the Luke variational principle for steady periodic gravity waves

Consider a steady two-dimensional irrotational flow of an ideal incompressible fluid in an infinite λ-periodic domain bounded by a λ-periodic line $y=\eta(x)$ from above and by a horizontal bottom $y=0$ from below. One period Ω_z of such a flow is shown in figure 1(*a*). The function $\eta(x)$ satisfies the conditions

$$\eta(x)>0, \quad \eta(x+\lambda)=\eta(x). \tag{2.1a,b}$$

Let $\phi(x,y)$ be the potential of the flow. For the harmonic function $\phi(x,y)$, we have the following boundary conditions

$$\left[\frac{\partial\phi}{\partial n}\right]_{y=\eta(x)}=0, \quad \left[\frac{\partial\phi}{\partial n}\right]_{y=0}=-\left[\frac{\partial\phi}{\partial y}\right]_{y=0}=0, \tag{2.2a,b}$$

where $\boldsymbol{n}$ is the unit outer normal to the boundaries of the flow domain. The velocity vector $\boldsymbol{q}$ and its components u, v are

$$\boldsymbol{q}=\nabla\phi, \quad u=\frac{\partial\phi}{\partial x}, \quad v=\frac{\partial\phi}{\partial y}, \tag{2.3a–c}$$

where ∇ is the nabla operator. The volume flux Q is

$$Q=\int_0^{\eta(x)} u(x, y)\,\mathrm{d}y. \tag{2.4}$$

We assume that $Q>0$, i.e. the fluid moves from left to right.

In the flow domain, the components of the velocity vector $u(x, y)$, $v(x, y)$ are λ-periodic functions with respect to the variable x, but the potential $\phi(x, y)$ increases by an increment C (by a circulation) on every period:

$$\phi(x+\lambda, y)-\phi(x, y)=C. \tag{2.5}$$

For a fixed upper boundary the flow is defined uniquely by specifying the circulation C.

Consider now a system of two-dimensional irrotational periodic waves of wavelength λ propagating with phase velocity c under the influence of gravity g without change of form from right to left on the surface of a fluid of finite depth. In the wave-fixed frame of reference the flow appears to be steady. Hence, we have a λ-periodic flow which moves from left to right, and its potential $\phi(x, y)$ satisfies the boundary conditions (2.2).

In addition to (2.2), on the unknown free surface $y=\eta(x)$ the Bernoulli equation should be fulfilled:

$$\left[\tfrac{1}{2}q^2+gy\right]_{y=\eta(x)}=R, \tag{2.6}$$

where $q=|\nabla\phi|$ is the modulus of the velocity vector $\boldsymbol{q}$ and R is the Bernoulli constant. Thus, the steady gravity wave problem is fully determined by the boundary conditions (2.2), (2.6) at specified values of λ, C and R.

For the λ-periodic flows, consider the Luke Lagrangian, which we take as the integral of pressure over one period (see Luke 1967):

$$L=\frac{1}{\rho}\int_0^{\lambda}\mathrm{d}x\int_0^{\eta(x)} p(x, y)\,\mathrm{d}y, \tag{2.7}$$

where ρ is the density of the fluid, p is the pressure. At fixed R and C by virtue of the Bernoulli equation,

$$\frac{p}{\rho}=R-\frac{q^2}{2}-gy, \tag{2.8}$$

so the functional L depends only on $\eta(x)$ because $q(x, y)=|\nabla\phi|$ is fully determined by the boundary condition (2.2). Hence, we can write

$$L[\eta]=\int_0^{\lambda}\mathrm{d}x\int_0^{\eta(x)}\left(R-\frac{1}{2}|\nabla\phi|^2-gy\right)\mathrm{d}y. \tag{2.9}$$

Varying the functional $L[\eta]$ at a fixed R, we obtain

$$\delta L[\eta]=\int_0^{\lambda}\left(R-\frac{1}{2}q^2[x, \eta(x)]-g\eta(x)\right)\delta\eta(x)\,\mathrm{d}x-\iint_{\Omega_z}\nabla(\delta\phi)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y, \tag{2.10}$$

where $\delta\phi(x, y)$ is the dependent variation of the potential $\phi(x, y)$ corresponding to the independent variation $\delta\eta(x)$.

Applying Green's first identity to the second integral in (2.10) and taking into account that the function $\phi(x, y)$ is harmonic, we get

$$\iint_{\Omega_z} \nabla(\delta\phi)\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega_z} \delta\phi \frac{\partial\phi}{\partial n}\,\mathrm{d}s, \tag{2.11}$$

where $\partial\Omega_z$ is the boundary of Ω_z, $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega_z$, ds is the line element along $\partial\Omega_z$. By virtue of (2.2), the contour integrals over $y=\eta(x)$ and $y=0$ vanish. Further, the integrals along the vertical lines $x=0$ and $x=\lambda$ cancel each other. Indeed, because the circulation C is fixed,

$$\delta\phi(\lambda, y) - \delta\phi(0, y) = \delta C = 0. \tag{2.12}$$

Besides, the function $u(x, y)$ is λ-periodic with respect to x, and on the vertical lines, we have

$$\left[\frac{\partial\phi}{\partial n}\right]_{x=0} = -u(0, y), \quad \left[\frac{\partial\phi}{\partial n}\right]_{x=\lambda} = u(\lambda, y) = u(0, y). \tag{2.13a,b}$$

Thus, the variation

$$\delta L[\eta] = \int_0^\lambda \left\{R - \frac{1}{2}q^2[x, \eta(x)] - g\eta(x)\right\}\delta\eta(x)\,\mathrm{d}x \tag{2.14}$$

under the constrain that C is fixed. It follows from (2.14) that for λ-periodic flows with a fixed circulation C, the dynamic boundary condition (2.6) will be satisfied if and only if $\delta L[\eta]$=0.

It is to be noted that for any λ-periodic flow

$$\iint_{\Omega_z} q^2(x, y)\,\mathrm{d}x\,\mathrm{d}y = CQ \tag{2.15}$$

(see, e.g. paper 1, p. 297).

Let $z = x + \mathrm{i}y$ be the complex variable. In the plane $w = \phi + \mathrm{i}\psi$, consider the rectangle Ω_w of length C and width Q, which corresponds to the flow period Ω_z in the physical z-plane (see figure 1a,b). By means of the function

$$\zeta = r_0 \exp\left(-\frac{2\pi w \mathrm{i}}{C}\right), \quad \text{where } r_0 = \exp\left(-\frac{2\pi Q}{C}\right), \tag{2.16}$$

we map conformally Ω_w onto an annulus with an outer radius of unity and inner radius of r_0 (see figure 1b,c). It follows from (2.16) that

$$Q = -\frac{C}{2\pi}\log r_0. \tag{2.17}$$

Taking into account (2.9), (2.15) and (2.17), we come to the formula

$$L[\eta] = \int_0^\lambda \mathrm{d}x \int_0^{\eta(x)} (R - gy)\,\mathrm{d}y + \frac{c_a^2\lambda^2}{4\pi}\log r_0, \tag{2.18}$$

where

$$c_a = \frac{C}{\lambda} = \frac{1}{\lambda}\int_0^\lambda u(x, y)\,\mathrm{d}x \tag{2.19}$$

is the averaged horizontal component of velocity at any level $y = \text{const.}$ completely within the fluid.

As in Longuet-Higgins (1975) and many other works, we non-dimensionalize all flow parameters by choosing $\lambda/(2\pi)$, $\sqrt{g\lambda/(2\pi)}$ and ρ as the scales for length, velocity and density, respectively. In what follows, all physical quantities will be dimensionless with accordance to the chosen scales. Now, the dimensionless wavelength is 2π, $\rho = g = 1$, the potential ϕ, circulation C and flux Q are scaled by $\sqrt{g\lambda^3/(2\pi)^3}$. The relation between Q and c_a is

$$Q = -c_a \log r_0. \tag{2.20}$$

The Luke functional L, scaled by $g\lambda^2/(2\pi)^2$, takes the form (for convenience, we double the functional and divide it by λ)

$$L[\eta] = 2Rh_a - H_a^2 + c_a^2 \log r_0, \tag{2.21}$$

where h_a and H_a are the mean and root-mean-square depths:

$$h_a = \frac{1}{2\pi}\int_0^{2\pi} \eta(x)\,\mathrm{d}x, \quad H_a^2 = \frac{1}{2\pi}\int_0^{2\pi} \eta^2(x)\,\mathrm{d}x. \tag{2.22a,b}$$

Any 2π-periodic domain can be transformed into a doubly connected one by means of the conformal mapping $t = \exp(\mathrm{i}z)$. The conformal invariant M, that is defined as the ratio r_1/r_2 $(r_1 > r_2)$ of the radii of an annulus onto which the doubly connected domain can be mapped, is called the modulus of the domain (see Bergman 1950, p. 102). Taking into account that in our case $M = 1/r_0$, one can see that the Luke Lagrangian L in the form (2.21) contains only geometric characteristics of 2π-periodic domains: the mean depths and the modulus. In terms of these geometric characteristics, the variational principal can be formulated in the form of the following proposition.

PROPOSITION 1. *A* 2π*-periodic domain is that of steady gravity waves if and only if*

$$\delta L[\eta] = \delta(2R\,h_a - H_a^2 + c_a^2 \log r_0) = 0, \tag{2.23}$$

where the parameters R *and* c_a^2 *are fixed.*

The parameter r_0 is responsible for the wave depth-to-length ratio and ranges between 0 and 1. For the waves of infinite depth r_0=0, for the solitary waves $r_0 = 1$. Application of the functional (2.21) is especially convenient when the parameter r_0 is given, as for example, in computations by Schwartz (1974) and Cokelet (1977). Indeed, in this case, at fixed c_a the variation $\delta(c_a \log r_0) = 0$, and the functional $L[\eta]$ is equivalent to

$$L_M[\eta] = 2R\,h_a - H_a^2. \tag{2.24}$$

So we can formulate the following proposition.

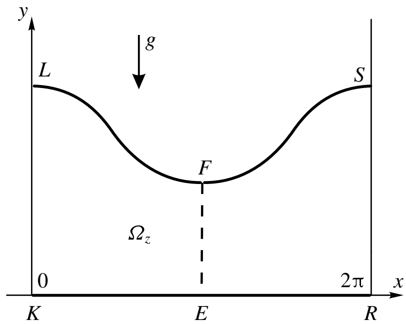

FIGURE 2. One wave period in the physical z-plane.

PROPOSITION 2. *In the class of* 2π*-periodic domains with the same modulus* $M = 1/r_0$*, the domain of periodic gravity waves is that for which*

$$\delta L_M[\eta] = 0, \tag{2.25}$$

where in (2.24) the parameter R is fixed.

We consider Propositions 1 and 2 as a geometric interpretation of the Luke variational principle.

3. System of quadratic equations for the Stokes coefficients

Let us assume that the waves are symmetric and the axis of ordinate is that of symmetry which goes through one of the wave crests. One period of the flow is shown in figure 2. We shall seek the conformal mapping of the annulus (figure 1*c*) in the parametric ζ-plane onto the flow domain of one wave period Ω_z in the form

$$z(\zeta) = 2\pi + \mathrm{i}\log\zeta - \mathrm{i}\log r_0 + \mathrm{i}\sum_{n=1}^{\infty} y_n\left(\zeta^n - \frac{r_0^{2n}}{\zeta^n}\right). \tag{3.1}$$

The representation (3.1) is a variant of the second Stokes method, suggested by Schwartz (1974). By virtue of the wave symmetry, the Stokes coefficients y_n ($n = 1, 2, 3\ldots$) are real. Moreover, for the representation (3.1) the condition on the bottom $\mathrm{Im} z = 0$ is automatically fulfilled. The parametric equations of the free surface are

$$x_s(\gamma) = 2\pi - \gamma - \sum_{n=1}^{\infty} \alpha_n y_n \sin n\gamma, \quad y_s(\gamma) = -\log r_0 + \sum_{n=1}^{\infty} \beta_n y_n \cos n\gamma, \tag{3.2a,b}$$

where $\alpha_n = 1 + r_0^{2n}$, $\beta_n = 1 - r_0^{2n}$, γ is the polar angle in the ζ-plane.

It is to be noted that the mapping (3.1) specifies a family of 2π-periodic domains with the same modulus $M = 1/r_0$. Thus, we are under the conditions of Proposition 2. In appendix A to the paper, it will be proved that

$$h_a = \tfrac{1}{2}\Lambda - \log r_0, \quad H_a^2 = \tfrac{1}{2}(\Lambda_1 + G) - \Lambda\log r_0 + \log^2 r_0, \tag{3.3a,b}$$

where

$$\Lambda = \sum_{n=1}^{\infty} n\beta_{2n} y_n^2, \quad \Lambda_1 = \sum_{n=1}^{\infty} \beta_n^2 y_n^2, \quad G = \sum_{k=2}^{\infty} y_k \sum_{n=1}^{k-1} \Gamma_{k-n,n} y_n y_{k-n}, \tag{3.4a–c}$$

$$\beta_n = 1 - r_0^{2n}, \quad \Gamma_{m,n} = m\beta_n\beta_{2m+n} + n\beta_m\beta_{m+2n}. \tag{3.5a,b}$$

Inserting (3.3) into (2.24) yields

$$L_M = (R + \log r_0)\Lambda - \tfrac{1}{2}(\Lambda_1 + G) - \log^2 r_0 - 2R\log r_0. \tag{3.6}$$

It follows from (3.4) and (3.5) that L_M is a function of the Stokes coefficients y_n ($n = 1, 2, 3, \ldots$) and the parameter r_0. In accordance with (2.25) of Proposition 2, we variate L_M with respect to y_n at fixed r_0 to get

$$(n\beta_{2n}K - \beta_n^2)y_n = \frac{1}{2}\frac{\partial G}{\partial y_n}, \quad n = 1, 2, 3, \ldots, \tag{3.7}$$

where

$$K = 2(R + \log r_0). \tag{3.8}$$

In appendix A to the paper, we shall prove that

$$\frac{\partial G}{\partial y_n} = \sum_{m=1}^{n-1} \Gamma_{n-m,m} y_{n-m} y_m + 2\sum_{m=1}^{\infty} \Gamma_{m,n} y_{m+n} y_m. \tag{3.9}$$

With allowance for (3.9) we deduce

$$(n\beta_{2n}K - \beta_n^2)y_n = \frac{1}{2}\sum_{m=1}^{n-1} \Gamma_{n-m,m} y_{n-m} y_m + \sum_{m=1}^{\infty} \Gamma_{m,n} y_{m+n} y_m, \quad n = 1, 2, 3, \ldots \tag{3.10}$$

and at $n = 1$ the first sum in (3.10) vanishing. The system (3.10) is an infinite system of quadratic equations with respect to the Stokes coefficients y_n.

Equation (2.23) of Proposition 1 allows us to express c_a^2 in terms of the Stokes coefficients. Indeed, fixing y_n and varying only r_0, we obtain

$$c_a^2 = -r_0\frac{\partial L_M}{\partial r_0} = K - \Lambda + \frac{r_0}{2}\left(\frac{\partial \Lambda_1}{\partial r_0} + \frac{\partial G}{\partial r_0} - K\frac{\partial \Lambda}{\partial r_0}\right), \tag{3.11}$$

where

$$\frac{\partial \Lambda}{\partial r_0} = \sum_{n=1}^{\infty} n\frac{\mathrm{d}\beta_{2n}}{\mathrm{d}r_0} y_n^2, \quad \frac{\partial \Lambda_1}{\partial r_0} = \sum_{n=1}^{\infty} 2\beta_n\frac{\mathrm{d}\beta_n}{\mathrm{d}r_0} y_n^2, \tag{3.12a,b}$$

$$\frac{\partial G}{\partial r_0} = \sum_{k=2}^{\infty} y_k \sum_{n=1}^{k-1} \frac{\mathrm{d}\Gamma_{k-n,n}}{\mathrm{d}r_0} y_n y_{k-n}. \tag{3.13}$$

The conformal mapping of the annulus Ω_ζ onto the rectangle Ω_w in the w-plane (see figure 1b,c) is determined by the equation

$$w = c_a\mathrm{i}(\log\zeta - \log r_0 - 2\pi\mathrm{i}). \tag{3.14}$$

If the depth of the fluid is infinite, then $r_0 \to 0$, $\beta_n = 1$, $\Gamma_{m,n} = m + n$, and the system (3.10) becomes simpler:

$$(nK - 1)y_n = \frac{n}{2}\sum_{m=1}^{n-1} y_{n-m}y_m + \sum_{m=1}^{\infty}(m+n)y_{m+n}y_m, \quad n = 1, 2, 3, \ldots \tag{3.15}$$

Equation (3.11) transforms to the relation

$$c_a^2 = K - \Lambda. \tag{3.16}$$

If $r_0 \to 0$, then $\log r_0 \to -\infty$, and the formula of the conformal mapping (3.1) loses its sense. To avoid this, it is enough to transfer the x-axes of the z-plane onto the mean level of waves by setting $z_1 = z - \mathrm{i}h_a$. It follows from (3.3a) that $\log r_0 = \Lambda/2 - h_a$, and then for the infinite depth,

$$z_1(\zeta) = 2\pi + \mathrm{i}\log\zeta - \mathrm{i}\frac{\Lambda}{2} + \mathrm{i}\sum_{n=1}^{\infty} y_n\zeta^n. \tag{3.17}$$

Thus, for the fixed parameters $K = 2(R + \log r_0)$ and r_0, any non-trivial solution to system (3.10) and formulae (3.1), (3.14) define a 2π-periodic wave domain and the parametric equations for the complex potential w.

We should remark that Schwartz (1974), and later Cokelet (1977), located the x-axes in the physical plane at the level d above the bottom, defining d as the depth of uniform stream flowing with speed c_a whose mass flux equals that of the wave. Hence, $d = Q/c_a$. Since $Q = -c_a \log r_0$, it is clear that $d = -\log r_0$. This elucidates the physical sense of the parameter K in system (3.10). Indeed, by virtue of (2.6) and (3.8), on the free surface we have

$$[q^2 + 2(y - d)]_{y=\eta(x)} = K. \tag{3.18}$$

This means that the parameter K is simply the Bernoulli constant as it was defined and designated in Schwartz (1974) and Cokelet (1977). If in solving system (3.10) we specify another parameter instead of K, for example, y_1 as in Stokes (1880), then (3.10) is a pure quadratic system of equations with respect to the unknowns K and y_n.

4. Algorithm for solving system (3.10)

Let a be the amplitude of the wave (one half the vertical distance from the crest to the trough). The goal of this section is to develop an algorithm for finding expansions for the Stokes coefficient y_n in powers of the wave amplitude a with coefficients depending on the mean depth h_a.

Because in system (3.10) the coefficients β_n and $\Gamma_{m,n}$ depend only on r_0^2, it is convenient to introduce

$$r = r_0^2, \quad \text{and hence,} \quad \beta_n = 1 - r^n. \tag{4.1a,b}$$

Now in (3.10), the coefficients β_n, $\Gamma_{m,n}$ depend on r, formula (3.11) for c_a^2 transforms to

$$c_a^2 = K - \Lambda + r\left(\frac{\partial\Lambda_1}{\partial r} + \frac{\partial G}{\partial r} - K\frac{\partial\Lambda}{\partial r}\right), \tag{4.2}$$

and in (3.12), (3.13) we need to replace r_0 by r.

At the first step, we follow Stokes (1880), namely, we assume that $y_1 = b$, where b is a specified small parameter. Since y_1 is given, the parameter K is now unknown. We shall seek the coefficients y_n $(n = 2, 3, \ldots)$ and the parameter K in the form of expansions in powers of b. Let an odd number N be the greatest power in these expansions, then

$$K = \sum_{n=0}^{(N-1)/2} K_{2n} b^{2n} + O(b^{N+1}), \quad y_i = \sum_{n=0}^{[(N-i)/2]} b_{i,2n} b^{i+2n} + o(b^N), \quad i = 2, 3, \ldots, N, \tag{4.3a,b}$$

where [ ] denotes the integer part of a real number, and the coefficients K_{2n} and $b_{i,2n}$ are to be found. The general number of unknown coefficients is $(N+1)^2/4$. The procedure of their consecutive search appears as follows.

(i) We substitute the expressions for K and y_i from (4.3) in the first N equations of system (3.10) and expand each of the equations into the series in powers of b up to the power N inclusively. We obtain N equations of the form $P_i(b) = 0$, $i = 1, 2, \ldots, N$, where $P_i(b)$ are N-degree polynomials in b. The coefficients of these polynomials depend on the unknowns K_{2n} and $b_{i,2n}$. Equating these coefficients to zero, we obtain a system of $(N+1)^2/4$ equations with respect to $(N+1)^2/4$ unknowns K_{2n} and $b_{i,2n}$. It is to be noted that if N is chosen to be even, then the number of unknowns will be greater by one than the number of equations, and not all coefficients will be found.

(ii) We organize the external loop for $j = 1, 2, \ldots, N$, where j is the power of the parameter b labelling the coefficients at b^j in the polynomials $P_i(b)$. Let the function $\lceil x \rceil$ designate a so called 'ceiling', i.e. the least integer greater than or equal to x. The unknown coefficients K_{2n} and $b_{i,2n}$ are determined by portions with $\lceil j/2 \rceil$ coefficients in each portion. In the external loop over j, we organize the internal loop for $k = 1, 2, \ldots, \lceil j/2 \rceil$. For each k we find

$$i_k = j - 2\lceil j/2 \rceil + 2k. \tag{4.4}$$

We equate the coefficient at b^j in the polynomial $P_{i_k}(b)$ to zero and from the obtained equation we find either

$$K_{j-1} \text{ (if } j \text{ is odd and } k = 1) \quad \text{or} \quad b_{i_k, j-i_k} \text{ (if } j \text{ is even or } k \neq 1), \tag{4.5a,b}$$

and in so doing all equations for finding K_{j-1} or $b_{i_k,j-i_k}$ being linear.

Formulae (4.4), (4.5) define exactly the consecutive procedure of determining K_{2n}, $b_{i,2n}$, and for each pair (j, k), allow us to express a new unknown coefficient in terms of the coefficients which have been already found. Note, by the way, that as follows from (4.5a), if N is even, then K_{2N} remains indefinite.

The above algorithm results in the expansions (4.3) with coefficients which are rational functions of $r = r_0^2$. Inserting these expansions into (4.2) gives

$$c_a^2 = \sum_{n=1}^{(N-1)/2} c_{2n} b^{2n} + O(b^{N+1}), \tag{4.6}$$

where the coefficients c_{2n} are also rational functions of r.

The second step is to express the parameter b in terms of the amplitude a and to reconstruct the expansions (4.3) and (4.6) in powers of a. It follows from (3.2b) that

$$a - \sum_{n=1}^{(N+1)/2} \beta_{2n-1} y_{2n-1} = 0. \tag{4.7}$$

Taking into account that the coefficients y_{2n-1} contain only odd powers of b, we shall seek the parameter b as

$$b = \sum_{n=1}^{(N+1)/2} b_{2n-1} a^{2n-1}. \tag{4.8}$$

Substituting (4.8) into (4.7) and expanding the left-hand side of (4.7) in powers of a up to the power N inclusively, we obtain an N-degree polynomial in odd powers of a with $(N+1)/2$ coefficients depending on the parameter r and the unknowns b_{2n-1}. Equating these coefficients to zero, we sequently find $b_1, b_3, \ldots, b_N$, which again turn out to be rational functions of r. Now we reconstruct the expansions (4.3) and (4.6) in powers of a by inserting in these expansions the expression (4.8) for b. As a result we have

$$K = \sum_{n=0}^{(N-1)/2} K'_{2n} a^{2n} + O(a^{N+1}), \quad c_a^2 = \sum_{n=1}^{(N-1)/2} c'_{2n} a^{2n} + O(a^{N+1}), \tag{4.9a,b}$$

$$y_i = \sum_{n=0}^{[(N-i)/2]} b'_{i,2n} a^{i+2n} + o(a^N), \quad i = 1, 2, \ldots, N, \tag{4.10}$$

where the coefficients K'_{2n}, c'_{2n} and $b'_{i,2n}$ are rational functions of $r = r_0^2$.

The third and last step is to express the parameter $r = r_0^2$ in terms of the mean depth h_a and to reconstruct the expansions (4.9) and (4.10) so that they would be in powers of a with coefficients depending h_a. Let $\varepsilon = \exp(-2h_a)$. It follows from (3.3$a$) and (3.4$a$) that

$$r - \varepsilon \exp\left(\sum_{n=1}^{(N-1)/2} n \beta_{2n} y_n^2 \right) = 0. \tag{4.11}$$

We shall seek the dependence between r and ε in the form

$$r = \varepsilon \left(1 + \sum_{n=1}^{(N-1)/2} \nu_{2n} a^{2n} \right), \tag{4.12}$$

where ν_{2n} are unknown coefficients depending on $\varepsilon = \exp(-2h_a)$.

Substituting (4.12) into (4.11) and expanding the left-hand side of (4.11) in powers of a up to the power $N-1$ inclusively, we obtain an $(N-1)$-degree polynomial in even powers of a with $(N-1)/2$ non-zero coefficients depending on the parameter ε and the unknowns ν_{2n}. The coefficient at a^0 in this polynomial is zero. Equating the remainder coefficients to zero, we sequently find $\nu_2, \nu_4, \ldots, \nu_{N-1}$, which are now rational functions of ε. Reconstruction of the expansions (4.9) and (4.10) by means of (4.12) yields

$$K=\sum_{n=0}^{(N-1)/2}\kappa_{2n}a^{2n}+O(a^{N+1}),\quad c_a^2=\sum_{n=1}^{(N-1)/2}\sigma_{2n}a^{2n}+O(a^{N+1}),\qquad(4.13a,b)$$

$$y_i=\sum_{n=0}^{[(N-i)/2]}a_{i,2n}a^{i+2n}+o(a^N),\quad i=1,2,\ldots,N,\qquad(4.14)$$

where the coefficients κ_{2n}, σ_{2n} and $a_{i,2n}$ are known rational functions of $\varepsilon=\exp(-2h_a)$.

This algorithm has been realized by means of Mathematica package which allows symbolic computations and exact arithmetics to be applied (see Wolfram 2003). In so doing, the routines Series, Coefficient and Solve have been used systematically. It is worth noticing that steps 1 and 2 of the algorithm correspond to the formulations by Schwartz (1974) and Cokelet (1977), when the depth $d=-\log r_0$ is fixed. The third step correspond to the formulations by De (1955) and Fenton (1985), when instead of the parameter $d=-\log r_0$, whose physical meaning is not quite clear, the mean wave depth h_a is specified.

We should remark that the first two steps can be programmed in such computer languages as Fortran or C++, but the most time-consuming third step requires necessarily the symbolic computations because in (4.11) we need to know the coefficients y_n as functions of r and a. Realizing all three steps, we have carried out the computations at $N=21$.

5. Expansions for the wave resistance and other wave properties in terms of the geometric wave parameters: h_a and a

Applying the algorithm of § 4, we obtain the expansions (4.13) and (4.14) in powers of a with coefficients which are rational functions of the parameter $\varepsilon=\exp(-2h_a)$. As we shall see in what follows, finding the expansion for the wave resistance requires the knowledge of the expansions for the mean potential energy V and the root-mean-square velocity c_b at the bottom in steady motion, which are defined as

$$V=\frac{1}{4\pi}\int_0^{2\pi}[\eta(x)-h_a]^2\,\mathrm{d}x,\quad c_b^2=\frac{1}{2\pi}\int_0^{2\pi}u^2(\xi,0)\,\mathrm{d}\xi.\qquad(5.1a,b)$$

It can be easily seen from the definitions of V and H_a^2 that $V=(H_a^2-h_a^2)/2$. This relation and equations (3.3) yield

$$V=\frac{1}{4}\left(\Lambda_1+G-\frac{\Lambda^2}{2}\right).\qquad(5.2)$$

Using (3.4) and (4.14), we get the expansion for V. In so doing, the coefficients β_n and Γ_{mn}, which depend on r, should be also expanded in powers of a by means of formula (4.12).

For the parameter c_b^2, it is possible to prove the identity

$$c_b^2+2h_a=2R\qquad(5.3)$$

(see equation (3.22) in paper 1). From this identity and equations (3.3*a*), (3.8) we conclude that

$$c_b^2 = K - \Lambda, \tag{5.4}$$

which allows us to find the expansion for c_b^2.

Implementation of the algorithm of § 4 and the above manipulations leads to expansions for the wave parameters c_a^2, c_b^2, V, Λ, which are needed to get the expansion for the wave resistance. All these expansions are in powers of a with coefficients which are rational functions of $\varepsilon = \exp(-2h_a)$. Besides, the expansions for c_a^2, c_b^2, V and Λ contain only even powers of a, moreover, they possess a very useful feature: the numerators of their coefficients (but not denominators) are so-called palindromic polynomials of ε. By definition, a polynomial $P_n(\varepsilon)$ of degree n is palindromic if

$$P_n(\varepsilon) = \varepsilon^n P_n(1/\varepsilon). \tag{5.5}$$

This feature allows us to shorten representations of polynomials by one-half, for example,

$$P_2(2, 3) = 2 + 3\varepsilon + 2\varepsilon^2, \quad P_3(2, 3) = 2 + 3\varepsilon + 3\varepsilon^2 + 2\varepsilon^3, \quad P_4(1, 0, 4) = 1 + 4\varepsilon^2 + \varepsilon^4, \tag{5.6a–c}$$

and so on.

In appendix B to the paper, formulae (B 1)–(B 4) present the expansions for c_a^2, c_b^2, Λ and V up to a^6 ($N = 7$). In spite of the fact that we have carried out the computations at $N = 21$, expansions for $N \geqslant 9$ become too long to be printed. For example, the formula for c_a^2 at $N = 21$ after conversion to LaTeX takes approximately 75 kilobytes, i.e. approximately 25 journal pages. For completeness, an expansion for the mean kinetic energy T_{c_a}, computed under the assumption that the waves propagate with the speed c_a in the bottom-fixed reference frame, is also presented by (B 5). The expansion for T_{c_a} can be easily obtained from (3.3*a*) and formula (5.21) from the paper by Cokelet (1977):

$$T_{c_a} = \tfrac{1}{2} c_a^2 (h_a + \log r_0) = \tfrac{1}{4} c_a^2 \Lambda. \tag{5.7}$$

In paper 1, we have derived formula (3.13) for the wave resistance R_w of a two-dimensional body that moves horizontally at a constant speed c in a channel of finite depth h. In dimensionless form, this formula can be written as

$$R_w = 3V + \tfrac{3}{2}\delta_1^2 + (h - c^2)\delta_1, \tag{5.8}$$

where $\delta_1 = h_a - h$ is the defect of levels (the difference between the mean level far downstream and the undisturbed level far upstream). The scale for R_w in (5.8) is $\rho g \lambda^2/(2\pi)^2$. The formula has been derived under assumptions that in the body-fixed reference frame the flow is steady and irrotational.

Assume that for the waves far downstream we know the mean depth h_a and amplitude a. The knowledge of these parameters is enough to compute all wave properties, but to find the wave resistance by means of (5.8), we should determine the parameters c and h. In a sense, this is an inverse problem: in the body-fixed reference frame find the speed c and depth h of a uniform stream which without

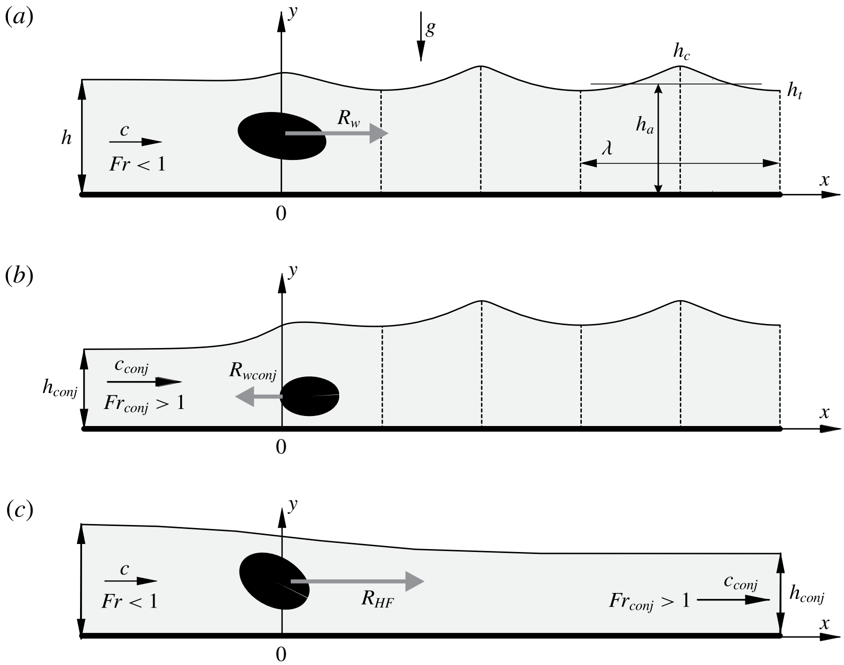

FIGURE 3. Sketches of subcritical flows (a), supercritical flows (b) and hydraulic falls (c).

dissipation is able to create a given system of waves far downstream due to some disturbance located in the stream.

By virtue of the fact that the Bernoulli constant R and the flow flux Q far upstream and far downstream are the same, the parameters c and h satisfy the following system of equations

$$\frac{c^2}{2} + h = R, \quad ch = Q, \tag{5.9a,b}$$

where R and Q are known wave properties. With the help of some results obtained in the works by Keady & Norbury (1975) and Benjamin (1995), it has been demonstrated in paper 1 that system (5.9) always has two solutions, (c_1, h_1) and (c_2, h_2). For the first solution, the Froude number $Fr = c_1/\sqrt{h_1} < 1$, for the second one, the Froude number $Fr = c_2/\sqrt{h_2} > 1$. So, for the first solution, the upstream flow is subcritical, for the second one, it is supercritical.

In this paper, for convenience, we rename the designations for these solutions. The first, subcritical solution, we designate simply as (c, h), and the second, supercritical one, we designate as (c_{conj}, h_{conj}), taking into consideration that two uniform streams whose speed c and depth h satisfy (5.9) with the same R and Q are called conjugate streams (see, e.g. Keady & Norbury 1975). Now, the subcritical Froude number is simply Fr, and supercritical one is Fr_{conj}. The sketches of the subcritical and supercritical flows are shown in figure 3(a,b). It is worthwhile to stress that for both cases the waves far downstream are the same, but the bodies which generate these waves can differ in shape, size and their location with respect to the bottom.

In paper 1, we have derived explicit analytical formulae (4.11), (4.12), (3.28*b,c*) for the solutions to system (5.9). In dimensionless form, the formulae can be rewritten as

$$h = \frac{2R}{Fr^2+2}, \quad c^2 = Fr^2 h, \tag{5.10a,b}$$

$$h_{conj} = \kappa h, \quad c^2_{conj} = \frac{Fr^2 h}{\kappa^2}, \quad Fr^2_{conj} = \frac{Fr^2}{\kappa^3}, \tag{5.11a–c}$$

where

$$\kappa = \frac{Fr^2}{4}\left(1+\sqrt{1+\frac{8}{Fr^2}}\right) < 1, \tag{5.12a}$$

$$Fr^2 = 6p \sin\left[\frac{1}{3}\arcsin\left(\frac{1}{p}\right)\right] - 2 < 1, \quad p = \sqrt{\frac{8R^3}{27Q^2}} > 1. \tag{5.12b,c}$$

5.1. *Analytical formulae for R_w in the subcritical case*

By making use of (5.8), (5.10), (5.12) and the expansions for c_a^2, c_b^2, $\varLambda$, V, the wave resistance for the subcritical case can be easily computed. Indeed, with the help of the identity (5.3) and equations (2.20), (3.3*a*), (5.8) (5.10*a*), we derive that

$$R_w = 3V - \frac{3}{2}\delta_1^2 + h_a(1-Fr_b^2)\delta_1, \quad \delta_1 = h_a\frac{Fr^2 - Fr_b^2}{Fr^2+2}, \quad p = \sqrt{\frac{(c_b^2+2h_a)^3}{27c_a^2(\varLambda/2-h_a)^2}}, \tag{5.13a–c}$$

where Fr_b is the Froude number based on the mean depth h_a and the root-mean-square bottom velocity c_b:

$$Fr_b = \frac{c_b}{\sqrt{h_a}}. \tag{5.14}$$

Now, if the geometric parameters of the waves a and h_a are given, we compute the parameter p by (5.13*c*), Fr^2 by (5.12*b*), Fr_b by (5.14), δ_1 by (5.13*b*) and, finally, R_w by (5.13*a*). In figure 4, we compare the results obtained by this analytical approach (long-dashed lines) with accurate numerical computations of R_w carried out by the method of Maklakov (2002) (solid lines). The long-dashed lines are plotted only for $h_a < \infty$, because for $h_a = \infty$ such a multistep technique of computing R_w cannot be applied. As one can see, if the waves are not very steep, then even for small depths the results are in good agreement. The variable of the abscissa axes for the graphs of figure 4 is the waves steepness a/π.

Application of formulae (5.12)–(5.14) for computing R_w leaves some feeling of dissatisfaction caused by the necessity of taking several steps before getting the final result for R_w. It seems that it would be better to obtain a direct asymptotic expansion of R_w in powers of a. A simple way to do this is to rewrite system (5.9) by the use of (2.20), (3.3*a*) and (5.3) in the form

$$c^2 - 2\delta_1 = c_b^2, \quad c(h_a - \delta_1) = c_a(h_a - \varLambda/2). \tag{5.15a,b}$$

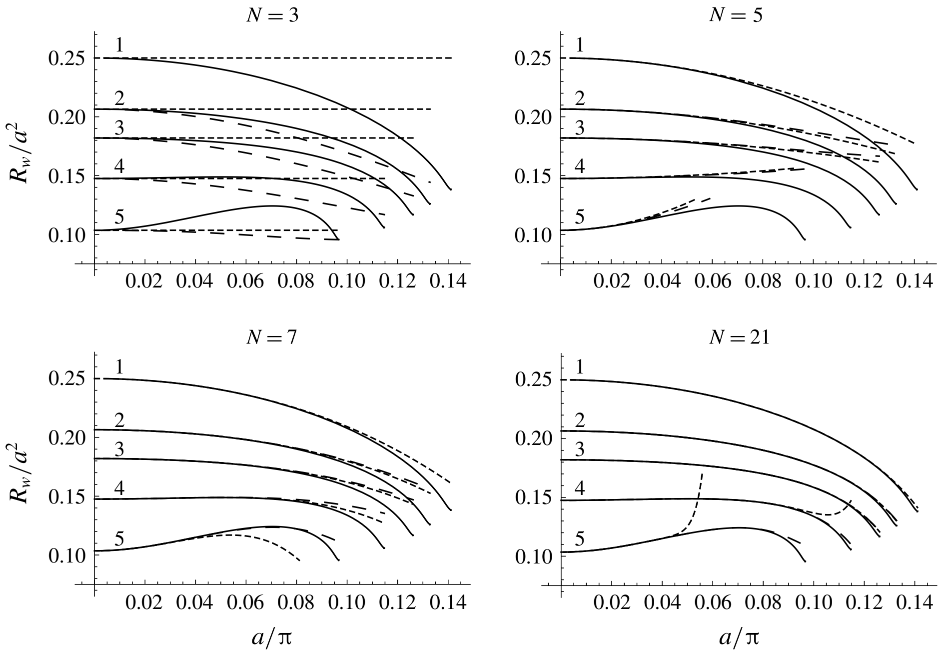

FIGURE 4. Comparison of analytical formulae with accurate numerical results for different N and $h_a/2\pi = \infty$, 0.3, 0.25, 0.2, 0.15 (1–5). Solid lines, numerical method of Maklakov (2002); long-dashed lines, analytical formulae (5.12)–(5.14); dashed lines, the direct asymptotic expansion of R_w in powers of a.

Excluding the unknown c from system (5.15), we obtain the following cubic equation with respect to δ_1:

$$(c_b^2 + 2\delta_1)(h_a - \delta_1)^2 - c_a^2(h_a - \Lambda/2)^2 = 0, \tag{5.16}$$

where the expansions for c_a^2, c_b^2 and Λ are known. To find an expansion for δ_1, we proceed in the same manner as in solving (4.8) or (4.11), namely, let

$$\delta_1 = \sum_{n=1}^{(N-1)/2} d_{2n} a^{2n}, \tag{5.17}$$

where d_{2n} are unknown coefficients. Substituting (5.17) into (5.16) and expanding the left-hand side of (5.16) in powers of a up to the power N inclusively, we obtain an $(N-1)$-degree polynomial in even powers of a with $(N-1)/2$ coefficients depending on the parameters $\varepsilon = \exp(-2h_a)$, h_a and the unknowns d_{2n}. The coefficient at a^0 in this polynomial is zero. Equating the remainder coefficients to zero, we sequently find $d_2, d_4, \ldots, \mathrm{d}_{N-1}$, which are now rational functions not only of ε but of ε and h_a. After determining the expansion for δ_1, we find the expansion for c by means of (5.15*a*). Inserting the obtained values of $h = h_a - \delta_1$ and c into (5.8), we find the expansion for the wave resistance R_w. Another way to do the same is to insert δ_1 directly into (5.13*a*).

It needs to be emphasized that the values of h and c, found in such a manner, correspond to the subcritical solution to the system (5.9). This is a consequence of the representation (5.17). Indeed, if the amplitude $a \to 0$, then we conclude from (5.17) that $\delta_1 \to 0$. In this case, as follows from (5.15*a*), $c^2 \to c_b^2$. Taking into account that, when $a \to 0$, the velocity c_a satisfies the ordinary dispersion relation of the linear theory and $c_a = c_b$, we find that

$$c^2 = \frac{1-\varepsilon}{1+\varepsilon} = \tanh h_a. \tag{5.18}$$

This means that the Froude number $Fr = c/\sqrt{h_a} < 1$ as $a \to 0$. Hence, the representation (5.17) corresponds to the subcritical branch of solutions to system (5.9).

Formulae (B 6) and (B 7) of appendix B present these direct asymptotic expansions for R_w and δ_1 up to a^6. If we take (B 6) only up to a^2, then

$$R_w = a^2 \left[\frac{h_a \varepsilon}{(\varepsilon - 1)(\varepsilon + 1)} + \frac{1}{4} \right] = \frac{a^2}{4} \left(1 - \frac{2h_a}{\sinh 2h_a} \right), \tag{5.19}$$

which exactly coincides with Kelvin's formula for R_w obtained from the linear wave theory (see Kelvin 1887). Formula (B 6) generalizes the deep water fourth-order result by Duncan (1983):

$$R_w = \tfrac{1}{4}a^2 - \tfrac{3}{8}a^4, \tag{5.20}$$

on to the case of finite depth up to a^6.

We should remark that the mixture of h_a and $\varepsilon = \exp(-2h_a)$ in the coefficients at a^n makes the expansions for R_w and δ_1 much longer than the previous ones when the coefficients depend only on $\varepsilon = \exp(-2h_a)$. For example, the expansion for R_w at $N = 21$ after conversion to LaTeX takes approximately 1300 kilobytes (compare with 75 kilobytes for c_a^2), which corresponds to approximately several hundreds of journal pages.

In figure 4, the dashed lines are the results of computations of R_w by its direct expansion in powers of a. As one can see, for small depths ($h_a/2\pi = 0.2, 0.15$) the results are worse than those obtained by the multistep method (long-dashed lines in figure 4). An explanation to this is the behaviour of the function $Fr^2(p)$ defined by formula (5.12*b*). Indeed, the same direct asymptotic expansion of R_w can be found by expanding the parameter p, defined by (5.13*c*), and after that, by expanding the function $Fr^2(p)$ in the Taylor series at the point p_0, where p_0 is the value of p at $a = 0$. The graph of the function $Fr^2(p)$ is shown in figure 5(*a*), and as one can see, near the point $p = 1$ the derivatives of $Fr^2(p)$ are very large. As can be seen from figure 5(*b*), for $h_a/2\pi = 0.2, 0.15$ the values of p_0 is very close to unity. So it is difficult to expect a good result in the case of shallow water for the direct expansion of R_w when we have such an unpleasant behaviour of $Fr^2(p)$ in the vicinity of $p = 1$.

The infinite depth case is also subcritical with the Froude number $Fr = 0 < 1$. For this case formulae (B 8)–(B 11) of appendix B present asymptotic expansions up to a^{21} for R_w, $c^2 = c_a^2 = c_b^2$, V and T. Dallaston & McCue (2010) presented the formula for c_a^2 to a^{14} (equation (11) of their paper). Comparison shows that all coefficients in their formula coincide with those of (B 9). In figure 4, the dashed lines labelled by number 1 demonstrate the results of computations of R_w by formula (B 8) in comparison with accurate numerical computations by the method of Maklakov (2002) (solid lines). As one can see, the coincidence is not so bad, especially at $N = 21$.

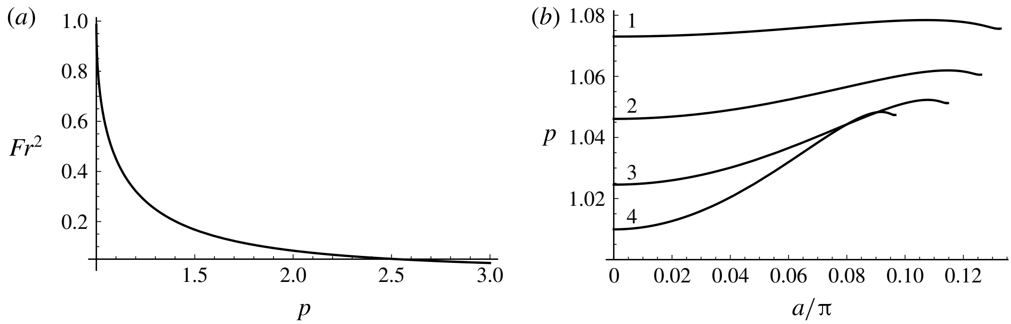

FIGURE 5. Graph of the function $Fr^2(p)$ and the dependence of p on a/π for $h_a/2\pi = 0.3$, 0.25, 0.2, 0.15 (1–4), computed by the numerical method of Maklakov (2002).

5.2. *Supercritical case*

Free-surface flows with periodic downstream waves generated by a supercritical stream past an obstacle were discovered by Dias & Vanden-Broek (2002), and further examples were computed by a number of authors (see, e.g. Dias & Vanden-Broek 2004; Binder, Vanden-Broek & Dias 2009). In figure 3(*a*,*b*), we demonstrate the sketches of subcritical and supercritical flows. For the supercritical case, the total pressure force exerted to the obstacle from the side of fluid we denote R_{wconj}. As has been rigorously proved in paper 1, the values of R_w (subcritical case) are always positive, but those of R_{wconj} (supercritical case) are negative, which makes doubtful the physical realizability of flows in figure 3(*b*). Now we want to estimate the contribution of periodic waves to these negative values.

Consider the dimensionless form of the wave parameter introduced by Benjamin & Lighthill (1954), namely, the flow force

$$S = \int_0^{\eta(x)} [u^2(x, y) + p]\,\mathrm{d}y. \tag{5.21}$$

It follows from the momentum equation that

$$R_w = S_u(c, h) - S, \quad R_{wconj} = S_u(c_{conj}, h_{conj}) - S, \tag{5.22a,b}$$

where $S_u(c, h)$ is the flow force S for a uniform stream with speed c and depth h. This leads to the formula

$$R_{wconj} = R_w - [S_u(c, h) - S_u(c_{conj}, h_{conj})]. \tag{5.23}$$

The difference in square brackets is just the wave resistance for a so-called hydraulic fall: a waveless free-surface flow which is subcritical far upstream and supercritical far downstream (see Forbes 1988; Shen & Shen 1990). The sketch of the hydraulic fall is shown in figure 3(*c*). We denote the wave resistance for such flows by R_{HF}, its value being given by formula (3.28*a*) of paper 1. Thus, we come to the equations

$$R_{wconj} = R_w - R_{HF}, \quad R_{HF} = \frac{h^2}{2}\frac{(1-\kappa)^3}{1+\varkappa}, \tag{5.24a,b}$$

where $\kappa = h_{conj}/h$ is defined by (5.12*a*). Since $\kappa < 1$, we have $R_{HF} > 0$, and according to the corollary to Proposition 1 of paper 1 (p. 299), $R_{HF} > R_w$. So, the contribution

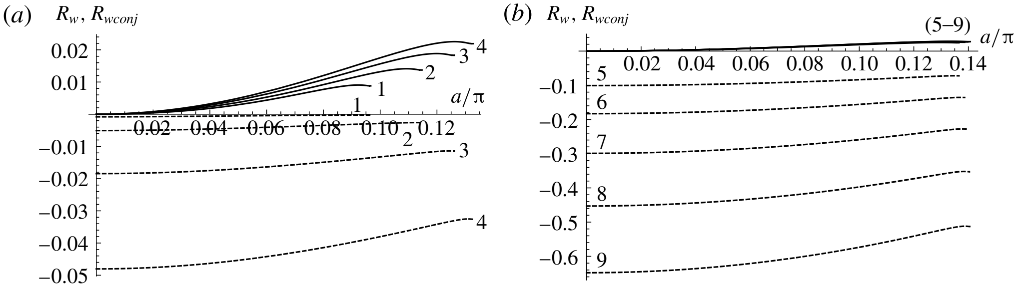

FIGURE 6. Dependences of R_w (solid lines) and R_{wconj} (dashed lines) on the wave steepness a/π for different depths $h_a/2\pi = 0.15$–0.55 with a step of 0.05 (1–9). Computations have been carried out by the method of Maklakov (2002).

of periodic wave train to the wave resistance is positive independently of the case, subcritical or supercritical. It is curious that all free flows shown in figure 3 are nonlinear, but their wave resistances are connected by the linear relationship (5.24*a*).

We do not give any expansions for the supercritical case due to our doubtfulness of its physical realizability, but for completeness we plot the graphs of R_w (solid lines) together with its negative counterpart R_{wconj} (dashed lines) in figure 6 for the depths $h_a/2\pi = 0.15$–0.55 with a step of 0.05. The computations have been carried out by the method of the paper of Maklakov (2002). As one can see from the graphs, as the depths increase, the moduli of $R_{wconj} < 0$ increase much faster than those of $R_w > 0$, creating a significant 'wave thrust'.

5.3. *Comparison with Fenton's fifth-order wave theory*

Fenton (1985) published a fifth-order wave theory based on the first Stokes method and presented explicit analytical expansions of wave properties in powers of a with coefficients depending on h_a. Since our expansions have the same structure but have been obtained by the second Stokes method, it is useful to compare the results of these independent calculations. Table 1 of Fenton's work contains 22 coefficients which take part in formulae for the potential $\phi(x, y)$, mean velocity c_a, shape of free surface $\eta(x)$, volume flux Q and Bernoulli constant R. These 22 coefficients have a rather complex structure and depend mainly on the parameter $S = \operatorname{sech}(2h_a)$ (do not confuse this S with the flow force, defined by (5.21)). Taking into account that

$$c_a = \sqrt{c_a^2}, \quad Q = c_a(h_a - \Lambda/2), \quad R = c_b^2/2 + h_a \qquad (5.25a\text{–}c)$$

and using (B 1)–(B 3), we have obtained the corresponding expansions and compared them with those presented by Fenton. The results turn out to be in full coincidence.

To compare the mean potential energy V, given by (B 4), we have used the explicit analytical expression for $\eta(x)$ of Fenton's work. By means of (5.1*a*), we have calculated V to a^4 and compared the result with (B 4). Again, we have had the same full coincidence.

6. Concluding remarks

In this paper, for steady periodic gravity waves in water on finite depth, we have derived a quadratic system of equations with respect to the coefficients of the second

Stokes method and developed an effective computer algorithm for solving the system. Making use of exact arithmetics and symbolic calculations, we have obtained explicit analytical expansions in powers of the wave amplitude a for the wave resistance and other wave properties to a^{21}. The coefficients of these expansions depend only on the mean wave depth h_a.

To our regret, the expansions of order higher than seven are too long to be printed, and in appendix B to the paper we have presented the results only up to a^7. Having taken a look at these formulae, the reader can easily imagine the length of expansions of order a^9 and higher. In spite of this evident deficiency, the formulae have also an important advantage: because even their LaTeX forms for printout have been generated by computer, they cannot contain any typos or calculational errors. On the other hand, these long analytical formulae, which sometimes take several hundreds of journal pages, are a logical consequence of accurate application of the famous analytical Stokes approach.

A usual way to enhance the convergence of the high-order Stokes expansions is the use of Padé approximants. However, this approach is effective when the Stokes coefficients are numbers, decimal as in Schwartz (1974) and Cokelet (1977) or rational as in Dallaston & McCue (2010). In our case, the coefficients are rational functions of $\varepsilon = \exp(-2h_a)$ (at best) or a mixture of h_a and ε (at worst). By specifying a and h_a, we are able to compute these coefficients and represent them as numbers with any quantity of decimal places, but then all analyticity of our approach will be lost and we return to the previous investigations.

We have compared our asymptotic expansions with those obtained by the fifth-order theory of Fenton (1985) and found that the results are in full coincidence. Comparison also has been made with the results obtained in paper 1, where the waves resistance R_w was computed by the accurate numerical method of the work by Maklakov (2002). The conclusion is as follows: analytical expansions to a^7 can be applied if the depth-to-wavelength ratio is not less than 0.15 and the wave amplitude is approximately 60 % of its limiting value.

All expansions used in the paper are stored in the supplementary materials and are available at https://doi.org/10.1017/jfm.2017.62 in the form of Mathematica expressions. The order of the stored expressions is 21. The materials contain the file 'Instruction.nb', which explains how to use the stored information.

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Supplementary materials

Supplementary materials are available at https://doi.org/10.1017/jfm.2017.62.

Appendix A. Proofs of formulae (3.3), (3.4) and (3.9)

First, we prove formula (3.3a). In terms of the Stokes coefficients the mean depth

$$h_a = \frac{1}{2\pi}\int_0^{2\pi}\left(-\log r_0 + \sum_{n=1}^{\infty}\beta_n y_n \cos n\gamma\right)\left(1 + \sum_{n=1}^{\infty} n\alpha_n y_n \cos n\gamma\right)\mathrm{d}\gamma. \tag{A 1}$$

Opening the brackets and integrating leads to (3.3a) and (3.4a).

Now, we prove (3.3*b*). For the root-mean-squared depth H_a we have

$$H_a^2 = \frac{1}{2\pi}\int_0^{2\pi}\left(-\log r_0 + \sum_{n=1}^{\infty}\beta_n y_n \cos n\gamma\right)^2\left(1+\sum_{n=1}^{\infty} n\alpha_n y_n \cos n\gamma\right)\,\mathrm{d}\gamma. \quad (\mathrm{A}\,2)$$

Again, we open the brackets and integrate to obtain

$$H_a^2 = \log^2 r_0 - \varLambda \log r_0 + \frac{\varLambda_1}{2} + \frac{G}{2}, \quad (\mathrm{A}\,3)$$

where $\varLambda_1$ is defined by (3.4*b*), and

$$G = \frac{1}{\pi}\int_0^{2\pi}\underbrace{\left(\sum_{n=1}^{\infty}\beta_n y_n \cos n\gamma\right)^2}_{f(\gamma)}\left(\sum_{n=1}^{\infty} n\alpha_n y_n \cos n\gamma\right)\,\mathrm{d}\gamma. \quad (\mathrm{A}\,4)$$

The first factor of the integrand in (A 4) we denote by $f(\gamma)$. Then

$$\begin{aligned} f(\gamma) &= \left(\sum_{n=1}^{\infty}\beta_n y_n \cos n\gamma\right)\left(\sum_{m=1}^{\infty}\beta_m y_m \cos m\gamma\right) \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\beta_n y_n \sum_{m=1}^{\infty}\beta_m y_m[\cos(n+m)\gamma + \cos(n-m)\gamma]. \end{aligned} \quad (\mathrm{A}\,5)$$

Representing $f(\gamma)$ as a Fourier series in cosine, we obtain

$$\begin{aligned} f(\gamma) &= \frac{\varLambda_1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\cos k\gamma\left(\sum_{m+n=k}\beta_n\beta_m y_n y_m + \sum_{n-m=k}\beta_n\beta_m y_n y_m\right. \\ &\quad \left. + \sum_{m-n=k}\beta_n\beta_m y_n y_m\right), \quad m, n = 1, 2, 3, \ldots, \end{aligned} \quad (\mathrm{A}\,6)$$

where for the sums in brackets, the lower equalities define the indexes which take part in forming the summands. Inserting this expression for $f(\gamma)$ in (A 4) and taking into account that

$$\sum_{n-m=k}\beta_n\beta_m y_n y_m = \sum_{m-n=k}\beta_n\beta_m y_n y_m, \quad k, m, n = 1, 2, 3, \ldots, \quad (\mathrm{A}\,7)$$

after integration we get

$$G = \sum_{k=1}^{\infty} k\alpha_k y_k\left(\frac{1}{2}\sum_{m+n=k}\beta_n\beta_m y_n y_m + A\right), \quad (\mathrm{A}\,8)$$

where

$$A = \sum_{k=1}^{\infty} k\alpha_k y_k \sum_{n-m=k}\beta_n\beta_m y_n y_m. \quad (\mathrm{A}\,9)$$

In the sum A, the index n changes from 1 to infinity, and at fixed n we have $m+k=n$. Now, we reorder the summation by forming the outer sum over n. This gives

$$A=\sum_{n=1}^{\infty}\beta_n y_n \sum_{m+k=n} k\alpha_k\beta_m y_k y_m=\sum_{k=1}^{\infty}\beta_k y_k \sum_{m+n=k} n\alpha_n\beta_m y_n y_m, \tag{A 10}$$

where to obtain the last equality we swap indexes k and n. Inserting this expression for A into (A 8) and taking into account that $k=m+n$, we infer

$$G=\sum_{k=2}^{\infty} y_k \sum_{m+n=k} \sigma_{m,n} y_m y_n, \tag{A 11}$$

where

$$\sigma_{m,n}=\tfrac{1}{2}(m+n)\alpha_{m+n}\beta_m\beta_n+n\alpha_n\beta_{m+n}\beta_m. \tag{A 12}$$

Here, the summation starts with $k=2$ since $m+n=k\geqslant 2$. To make the coefficients in the inner sum symmetric with respect to m and n, we introduce

$$\Gamma_{m,n}=(\sigma_{m,n}+\sigma_{n,m})/2. \tag{A 13}$$

Then

$$G=\sum_{k=2}^{\infty} y_k \sum_{m+n=k} \Gamma_{m,n} y_m y_n=\sum_{k=2}^{\infty} y_k \sum_{n=1}^{k-1} \Gamma_{k-n,n} y_n y_{k-n}, \quad \Gamma_{m,n}=\Gamma_{n,m}. \tag{A 14a,b}$$

After a little algebra, it is possible to demonstrate that $\Gamma_{m,n}$ is defined by equation (3.5b). This finalizes the proof of formulae (3.3b) and (3.4c). It is to be noted that reordering the summation in (A 9) is the key point for obtaining a compact expression for G.

Now we prove the equality (3.9). We denote

$$C_k=\sum_{m=1}^{k-1} \Gamma_{k-m,m} y_m y_{k-m}, \quad k=2,3,4,\ldots \tag{A 15}$$

It is easy to see that

$$\frac{\partial C_k}{\partial y_n}=\begin{cases}0 & \text{if } n>k-1 \Leftrightarrow k<n+1,\\ 2\Gamma_{k-n,n} y_{k-n} & \text{if } n\leqslant k-1 \Leftrightarrow k\geqslant n+1.\end{cases} \tag{A 16}$$

Now,

$$\frac{\partial G}{\partial y_n}=\sum_{m=1}^{n-1} \Gamma_{n-m,m} y_m y_{n-m}+2\sum_{k=n+1}^{\infty} \Gamma_{k-n,n} y_k y_{k-n}. \tag{A 17}$$

If we put in the second sum of (A 17) that $k-n=m$, we come to (3.9). It is worth noting that if $n=1$, then the first sum on the right-hand side of (3.9) vanishes.

Appendix B. Eleven asymptotical expansions for wave properties

The expansions below have been generated and converted into LaTeX by computer. For this reason, they cannot contain any typos or calculational errors. A principle of choosing the order of expansions has been very simple: we present an expansion only if any fraction in it takes not more than one line of the journal page. The notation $P_n(x_1, x_2, \ldots, x_{[n/2]+1})$ is that for palindromic polynomials of degree n in powers of ε, the arguments being the first half of coefficients.

B.1. *Expansions for finite depth with coefficients depending only on* $\varepsilon = \exp(-2h_a)$

$$c_a^2 = -\frac{a^6 P_{20}(9, 72, -2767, -2614, 114687, 491970, 1155180, 2005398, 2928016, 3608614, 3897270)}{6(\varepsilon-1)^{15}(\varepsilon+1)P_2(2,1)P_2(3,4)} + \frac{a^4 P_{10}(-1,-7,39,400,166,426)}{2(\varepsilon-1)^9(\varepsilon+1)} - \frac{a^2 P_4(1,0,16)}{(\varepsilon-1)^3(\varepsilon+1)} - \frac{\varepsilon-1}{\varepsilon+1}, \tag{B 1}$$

$$c_b^2 = -\frac{a^6 P_{20}(9, -132, -1954, 7640, 127743, 496104, 1139076, 2002512, 2912560, 3620356, 3886572)}{6(\varepsilon-1)^{15}(\varepsilon+1)P_2(2,1)P_2(3,4)} + \frac{a^4 P_{10}(-1,-3,71,296,302,290)}{2(\varepsilon-1)^9(\varepsilon+1)} - \frac{a^2 P_4(1,2,12)}{(\varepsilon-1)^3(\varepsilon+1)} - \frac{\varepsilon-1}{\varepsilon+1}, \tag{B 2}$$

$$\varLambda = -\frac{a^6(\varepsilon+1)P_{16}(-13,-56,671,2866,7542,10250,16461,16724,20710)}{2(\varepsilon-1)^{13}P_2(2,1)P_2(3,4)} + \frac{a^4(\varepsilon+1)P_2(1,-1)P_4(1,2,12)}{(\varepsilon-1)^7} - \frac{a^2(\varepsilon+1)}{\varepsilon-1}, \tag{B 3}$$

$$V = \frac{a^6 P_{16}(-19,-49,376,2061,5840,11203,16032,19281,20150)}{8(\varepsilon-1)^{12}P_2(2,1)P_2(3,4)} - \frac{a^4 P_2(1,4)P_4(1,0,4)}{8(\varepsilon-1)^6} + \frac{a^2}{4}, \tag{B 4}$$

$$T_{c_a} = \frac{a^6 P_{16}(-19,-25,108,473,4656,10679,17532,20313,22166)}{8(\varepsilon-1)^{12}P_2(2,1)P_2(3,4)} - \frac{3a^4\varepsilon P_4(1,-2,8)}{4(\varepsilon-1)^6} + \frac{a^2}{4}, \tag{B 5}$$

B.2. *Expansions for finite depth with coefficients which depend on a mixture of* $\varepsilon = \exp(-2h_a)$ *and* h_a

$$\begin{aligned}
R_w &= a^6\left[\frac{h_a^4\varepsilon(\varepsilon+1)^2P_{16}(-68, 135, 3756, 11729, 21080, 25063, 28084, 25953, 27736)}{4(\varepsilon-1)^{13}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^3}\right.\\
&\quad+\frac{h_a^3(\varepsilon+1)P_{18}(-19, -543, 1461, 29124, 91280, 158624, 198664, 220060, 223558, 229182)}{8(\varepsilon-1)^{12}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^3}\\
&\quad+\frac{h_a^2P_{18}(-51, -754, 3161, 42450, 131460, 225936, 288948, 329998, 351778, 364548)}{8(\varepsilon-1)^{11}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^3}\\
&\quad+\frac{h_aP_{18}(-39, -406, 3052, 27914, 82337, 142176, 187347, 221974, 244551, 255788)}{8(\varepsilon-1)^{10}(\varepsilon+1)P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^3}\\
&\quad\left.+\frac{P_{16}(-8, -67, 2264, 9339, 16832, 23481, 28248, 32623, 33776)}{16(\varepsilon-1)^9P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^3}\right]\\
&\quad+a^4\left[-\frac{h_a^2\varepsilon P_6(1, 10, -7, 10)}{(\varepsilon-1)^7(h_a\varepsilon+h_a+\varepsilon-1)}-\frac{h_aP_8(3, 14, 118, -10, 38)}{8(\varepsilon-1)^6(\varepsilon+1)(h_a\varepsilon+h_a+\varepsilon-1)}\right.\\
&\quad\left.-\frac{P_6(2, 1, 11, -10)}{4(\varepsilon-1)^5(h_a\varepsilon+h_a+\varepsilon-1)}\right]+a^2\left[\frac{h_a\varepsilon}{(\varepsilon-1)(\varepsilon+1)}+\frac{1}{4}\right],
\end{aligned} \tag{B 6}$$

$$\begin{aligned}
\delta_1 &= a^6\left[\frac{h_a^5\varepsilon(\varepsilon+1)^4P_{16}(-68, 135, 3756, 11729, 21080, 25063, 28084, 25953, 27736)}{4(\varepsilon-1)^{13}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^5}\right.\\
&\quad+\frac{h_a^4(\varepsilon+1)^3P_{18}(19, -197, 655, 15018, 49952, 61276, 65136, 29054, 33950, 8674)}{4(\varepsilon-1)^{12}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^5}\\
&\quad+\frac{h_a^3(\varepsilon+1)^2P_{18}(170, -75, 2588, 38862, 127206, 80752, -56918, -244606, -295670, -341418)}{8(\varepsilon-1)^{11}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^5}\\
&\quad+\frac{h_a^2(\varepsilon+1)P_{18}(270, 709, 960, 11402, 18362, -112344, -354978, -581050, -695942, -721978)}{8(\varepsilon-1)^{10}P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^5}\\
&\quad-\frac{h_aP_{18}(-140, -664, 2335, 14944, 56641, 163208, 316859, 452336, 532369, 553024)}{8(\varepsilon-1)^9P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^5}\\
&\quad\left.-\frac{(\varepsilon+1)P_{16}(50, -61, 3188, 10533, 27616, 44927, 63932, 70649, 76732)}{16(\varepsilon-1)^8P_2(2,1)P_2(3,4)(h_a\varepsilon+h_a+\varepsilon-1)^5}\right]\\
&\quad+a^4\left[-\frac{h_a^3\varepsilon(\varepsilon+1)^2P_6(1, 10, -7, 10)}{(\varepsilon-1)^7(h_a\varepsilon+h_a+\varepsilon-1)^3}-\frac{3h_a^2\varepsilon(\varepsilon+1)P_4(1, 14, 24)}{2(\varepsilon-1)^4(h_a\varepsilon+h_a+\varepsilon-1)^3}\right.\\
&\quad\left.+\frac{3h_aP_8(1, 0, 0, 32, 78)}{8(\varepsilon-1)^5(h_a\varepsilon+h_a+\varepsilon-1)^3}+\frac{(\varepsilon+1)P_6(3, 2, 13, 0)}{4(\varepsilon-1)^4(h_a\varepsilon+h_a+\varepsilon-1)^3}\right]\\
&\quad+a^2\left[\frac{h_a\varepsilon}{(\varepsilon-1)(h_a\varepsilon+h_a+\varepsilon-1)}-\frac{\varepsilon+1}{2(h_a\varepsilon+h_a+\varepsilon-1)}\right].
\end{aligned} \tag{B 7}$$

B.3. *Expansions for infinite depth*

$$\begin{aligned}
R_w &= \frac{a^2}{4}-\frac{3a^4}{8}-\frac{19a^6}{48}-\frac{2597a^8}{2880}-\frac{559733a^{10}}{201600}-\frac{9766021889a^{12}}{1016064000}\\
&\quad-\frac{1007540062126771a^{14}}{28165294080000}-\frac{141760453175028980\,2727a^{16}}{10149645374668800000}\\
&\quad-\frac{20636159768701655175517 24279a^{18}}{36575262072156487680000000}\\
&\quad-\frac{5237661911235593529828243406417 7671a^{20}}{22406444448547930230620160000000},
\end{aligned} \tag{B 8}$$

$$
\begin{aligned}
c^2=c_a^2=c_b^2 \;=\; & 1+a^2+\frac{a^4}{2}+\frac{a^6}{4}-\frac{22a^8}{45}-\frac{115069a^{10}}{25200}-\frac{6379033039a^{12}}{254016000}\\
& -\frac{875167353344611a^{14}}{7041323520000}-\frac{1509490905485738054687a^{16}}{253741134366720000}\\
& -\frac{2573919846675736709873575159a^{18}}{914381551803912192000000}\\
& -\frac{7438175790041291570074813078558143 1a^{20}}{56016111121369825576550400000000},
\end{aligned} \tag{B 9}
$$

$$
\begin{aligned}
V \;=\; & \frac{a^2}{4}-\frac{a^4}{8}-\frac{19a^6}{48}-\frac{3077a^8}{2880}-\frac{702113a^{10}}{201600}-\frac{12793118129a^{12}}{1016064000}\\
& -\frac{1366685377769731a^{14}}{28165294080000}-\frac{1981974995491975924247a^{16}}{1014964537466880000}\\
& -\frac{2965179893909091084219806119a^{18}}{3657526207215648768000000}\\
& -\frac{77199563300878281492652707790632631a^{20}}{2240644444854793023062016000000 0},
\end{aligned} \tag{B 10}
$$

$$
\begin{aligned}
T \;=\; & \frac{a^2}{4}-\frac{19a^6}{48}-\frac{3317a^8}{2880}-\frac{773303a^{10}}{201600}-\frac{14306666249a^{12}}{1016064000}\\
& -\frac{1546258035591211a^{14}}{28165294080000}-\frac{2264160227362818985007a^{16}}{1014964537466880000}\\
& -\frac{3415961852428553867553847039a^{18}}{3657526207215648768000000}\\
& -\frac{89611035395139454589837844653860111a^{20}}{22406444448547930230620160000000}.
\end{aligned} \tag{B 11}
$$

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